Length Dependence of the Period of a Pendulum

We can analyze the motion of a pendulum using forces and $\mathbf{F} = m\mathbf{a}$ or torques using $\tau = I\frac{d^2\theta}{dt^2}$. To handle a pendulum bob with finite spatial extent, the torque equation is more straightforward.

To see how it works, let's start with a simple pendulum, which is a point mass suspended from a fixed point. As the particle is pulled from the vertical by an angle θ, the gravitational torque tending to pull it back to a vertical position is

$$\tau = -Mg\sin\theta\, L,$$

where L is the distance from the point of suspension to (the center of mass of) the particle. Applying Newton's second law in the form $\tau = I\frac{d^2\theta}{dt^2}$ gives

$$I\frac{d^2\theta}{dt^2} = -MgL\sin\theta \qquad \longrightarrow \qquad \frac{d^2\theta}{dt^2} = -\frac{MgL}{I}\sin\theta$$

The moment of inertia I is the rotational analog of mass; it describes the extent to which an object resists a change in its state of rotational motion.

As discussed below, we can simplify this equation for small angles by replacing $\sin\theta \to \theta$, from which we deduce the period for small amplitudes

$$T = 2\pi\sqrt{\frac{I}{MgL}}$$

For a point particle, the moment of inertia I is just $I = ML^2$, but for an extended object, one can show that $I = I_0 + ML^2$, where I_0 is the moment of inertia for spinning about the center of mass of the object. This expression is called the **parallel axis theorem**. As L gets small, the term I_0 begins to cause a significant modification to the simple period expression. See pp. 303-5 of Serway for a more extensive discussion of the moment of inertia and the parallel axis theorem.

Amplitude Dependence of the Period of a Simple Pendulum

Starting from conservation of energy, you can show that

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{L}(\cos\theta - \cos\theta_0)$$

which can be separated to give

$$\sqrt{\frac{g}{L}}\,dt = \frac{d\theta}{\sqrt{2}\sqrt{\cos\theta - \cos\theta_0}}.$$

If we integrate the time taken for the pendulum to rise from the vertical ($\theta = 0$) to the maximum amplitude ($\theta = \theta_0$), the time taken is one quarter of the period. So

$$\frac{T}{4}\sqrt{\frac{g}{L}} = \int_0^{\theta_0} \frac{d\theta}{\sqrt{2}\sqrt{\cos\theta - \cos\theta_0}}$$

If the amplitude is small, so all angles in the path satisfy $\theta \ll 1$ (in radians), then we can approximate the cosine by

$$\cos\theta \approx 1 - \frac{\theta^2}{2!}.$$

You can show that this implies that for small amplitude oscillations, the integral on the right gives $\pi/2$, so that

$$\frac{T}{4}\sqrt{\frac{g}{L}} \approx \frac{\pi}{2} \qquad \longrightarrow \qquad T \approx 2\pi\sqrt{\frac{L}{g}}$$

As the amplitude gets larger, this approximation breaks down. Using the trignometric identity

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2},$$

you can rewrite the integral as

$$\int_0^{\theta_0} \frac{d\theta/2}{\sqrt{\sin^2\theta_0/2 - \sin^2\theta/2}}.$$

Now use the substitution $\sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\sin\phi$, where ϕ is the new independent variable. You can now simplify the right-hand side to

$$\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2\frac{\theta_0}{2}\sin^2\phi}},$$

which happens to be called the **complete elliptic integral of the first kind**.

Now use the binomial expansion,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

to turn the single integral into a bunch of integrals of the form $\int_0^{\pi/2}\sin^{2n}\phi\,d\phi$. Using

$$\int_0^{\pi/2} \sin^{2m}\theta\,d\theta = \frac{(2m-1)!!}{(2m)!!}\frac{\pi}{2},$$

where the double factorial is given by $n!! = n(n-2)(n-4)\cdots$. That is, $5!! = 5\times3\times1 = 15$ and $6!! = 6\times4\times2 = 48$, you can confirm the expansion

$$\boxed{T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \left(\frac{1!!}{2!!}\right)^2\sin^2\frac{\theta_0}{2} + \left(\frac{3!!}{4!!}\right)^2\sin^4\frac{\theta_0}{2} + \cdots\right]}$$